EXERCISES (CHERN CLASSES)

Two good references for Chern classes and vector bundles

(1) *Intersection Theory in Algebraic Geometry*, freely available at https://scholar.harvard.edu/files/joeharris/files/000-final-3264.pdf
(2) The all time classic *Characteristic classes* by Milnor and Stasheff

1. Vector bundles

Exercise 1 (Grassmannians)**.** *$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For $k \leq n$ the Grassmanian $\mathbf{Gr}(k, n)$ is defined to be the set*

$$\{E \subset \mathbb{K}^n \mid E \text{ is a } \mathbb{K} - \text{vector subspace of dimension } k\}$$

(1) Show that $\mathbf{Gr}(k, n)$ can be given the structure of a $k(n-k)$-manifold (real or complex depending on $\mathbb{K}$).
(2) Show that $\mathbf{Gr}(k, n)$ is compact.

Exercise 2. *Show that any vector bundle over a contractible manifold is trivial.*

Exercise 3. *Show that there are exactly two equivalence classes of real vector bundles over S^1, and one equivalence class of complex vector bundle over S^1.*

Exercise 4 (G-bundles)**.** *Let G be a subgroup of $\mathrm{GL}(n, \mathbb{K})$. A G-bundle is a vector bundle defined by an atlas of charts, the transition maps $U_i \cap U_j \times \mathbb{R}^n \longrightarrow U_i \cap U_j \times \mathbb{R}^n$ of which are of the form*

$$(x, v) \longrightarrow (x, g(x)v)$$

with $g(x) \in G$ for all x.

(1) A Euclidean bundle is a $\mathrm{O}(n)$-bundle. Show that any real vector bundle can be endowed with the structure of a Euclidean bundle.
(2) Show that any orientable 2-dimensional real bundle can be given the structure of a complex line bundle.

Exercise* 5. *Let E be a complex line bundle over a compact manifold M. Show that there exist a continuous map*

$$f : M \longrightarrow \mathbf{P}^m(\mathbb{C})$$

such that E is the pull-back of the tautological bundle over $\mathbf{P}^m(\mathbb{C})$ for a certain $m \in \mathbb{N}$.

2. First Chern class and line bundles

Exercise 6 (Classification of real line bundles)**.** *Let M be a manifold.*

(1) Let L be a (real) line bundle over M. Show that one can associate to L a cohomology class $a(L)$ in $\mathrm{H}^1(M, \mathbb{Z}/2\mathbb{Z})$ such that L is trivial if and only if $a(L) = 0$.
(2) Let L_1 and L_2 be two line bundles over M. Show that if $a(L_1) = a(L_2)$, L_1 and L_2 are isomorphic.

Exercise 7. *Let E_1 and E_2 be two line bundles over M. Show that*

$$c_1(E_1 \otimes E_2) = c_1(E_1) + c_1(E_2)$$

Exercise 8. *Compute the first Chern class of the tangent bundle a compact complex curve, using the triangulation method, and highlight how it relates to the Euler characteristic of the underlying real surface.*

Exercise 9. *Compute the first Chern class of the tautological bundle over $\mathbf{P}^1(\mathbb{C})$.*

Exercise 10 (Computing using sections)**.** *Let Σ be a smooth compact surface (a real 2-dimensional manifold) and let E be a smooth line bundle over Σ. A smooth section $s : \Sigma \longrightarrow E$ is called generic if the graph of s intersects the graph of the zero section transversally.*

Assume k is the algebraic intersection number of the graph of a generic section and 0_E the zero section. Show that

$$c_1(E) = k \cdot [\Sigma]$$

where $[\Sigma] \in \mathrm{H}^2(\Sigma, \mathbb{Z})$ is (a) fundamental class of Σ.

3. Higher dimensional Chern classes

Exercise 11. *Let E be a complex vector bundle that is the direct sum of line bundles*

$$E = \oplus_{i=1}^{m} L_i.$$

Show that $c_1(E) = \sum_{i=1}^{n} c_1(L_i)$.

Exercise 12. *Let E be complex vector bundle over X, and let L be a line bundle. Assume E is a direct sum of line bundles.*

For any r, compute $c_r(E)$ as a function of characteristic classes of E and L.

The formulae obtained in this exercise actually holds true by virtue of a general principle (sometimes referred to as *splitting principle*) that any formulae on Chern classes that holds true for bundles which are direct sums of line bundles is true in greater generality. We refer to *Intersection theory in Algebraic Topology* (https://scholar.harvard.edu/files/joeharris/files/000-final-3264.pdf), Chapter 5, Section 4.

Exercise 13 (Computing using sections II)**.** *Let E be an n-dimensional complex bundle over a compact n dimensional complex manifold X. Let s be a smooth section of E that intersects the zero section transversally. Show that $c_n(E)$ is equal to the algebraic intersection number between the graph of s and the graph of the zero section times the fundamental class of X*

Exercise 14. *Consider the following vector field on $\mathbb{C}^3$*

$$\vec{X} = \frac{1}{x+y+z}(yz\partial_x + xz\partial_y + xy\partial_z)$$

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(1) *Show that $\vec{X}$ is invariant under the action of $\mathbb{C}^*$ by multiplication.*
(2) *$\vec{X}$ thus passes to the quotient to $\mathbf{P}^2(\mathbb{C})$. Show that $\vec{X}$ vanishes in exactly 3 points.*
(3) *Show that the graph of the section $\vec{X}$ in $\mathrm{T}\mathbf{P}^2(\mathbb{C})$ intersects the zero section transversally.*
(4) *Show that $c_2(\mathbf{P}^2(\mathbb{C})) = 3 \cdot [\mathbf{P}^2(\mathbb{C}))]$.*